Transition Monoids of Multi-Wave Soliton Automata

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Abstract

Soliton automata are a type of molecular computers based on the predictable motions of standing energy waves, or solitons, propagating through a polymer molecule. Solitons are induced into the system and change the underlying covalent bond parities of the molecule as they move through it. Soliton automata have been proven to be computationally equivalent to the set of automata. While the behavior of a single soliton in a system is well-researched, the case of multiple solitons in a single system is not well-examined. In this work, we indicate some conclusions drawn from the definition of the model, notably the nature of determinism in such a model.

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Contents

1 Introduction

A *soliton*, or solitary wave, is a type of particle that passes through polymer chains that alter the underlying electron bond structure. These modifications can be considered as state changes in an automaton, working as an unconventional computer. We can model the molecular chain as a graph and describe the effect of solitons passing through it as a *soliton automaton*. Soliton automata were first examined by Dassow and Jürgensen in 1986 [2], where each atom in a molecule is considered a graph node, and where each bond is a graph edge. Each edge of the soliton graph has a weight designating a single or double bond. The soliton traverses the graph along edges of alternating weight; upon traversing an edge, the edge's weight is inverted. The propagation of solitons has been proven to be physically well-defined [6, 8].

Until recently, much of the work on soliton automata has focused on cases with only one soliton existing in an automaton at a time [2, 3, 4, 5, 7]. The *multi-wave soliton automata* model was initially examined by Bordihn, Jürgensen and Ritter in 2016 [1], but the paper left many open questions. In that model, a burst is defined as a sequence of (possibly delayed) soliton automata injections. In creating a parallel model of computation, questions arise regarding changes of the transition monoid of the soliton automata: Are the transformations induced by a burst potentially identities? Are they involutible? What variability can be permitted in the timing and positioning of bursts to achieve the same behavior in automata? Are there any impossible-to-traverse paths in a soliton graph? Is the nature of determinism in automata the same as in the single-soliton case? We intend to answer some of these questions in this thesis.

2 Preliminaries

2.1 Soliton Graphs

A *graph* is a pair $G = (N, E)$ with N the set of nodes and $E \subseteq N \times N$ the set of edges. A *weighted graph* is defined as the triple $G = (N, E, w)$ where (N, E) is a graph and $w(E)$ is the *weight function* mapping edges to integers. For a node $n \in N$, its *vicinity* is denoted by $V(n) = \{n' | (n, n') \in E\}$, and its *degree* is denoted by $d(n) = |V(n)|$.

A node is said to be *exterior* if $d(n) = 1$, and *interior* if $d(n) > 1$. A node is *isolated* if $d(n) = 0$.

A *soliton graph* [2] is a weighted graph $G = (N, E, w)$ such that:

1. $(n, n) \notin E$ for all $n \in N$;
2. Every component of G has at least one exterior node;
3. $1 \leq d(n) \leq 3$ for all $n \in N$;
4. If n is an exterior node, $w(n, n') \in \{1, 2\}$ for any node n' in the vicinity of n; and
5. For any node $n \in N$, if $d(n) \in \{2, 3\}$, then $\sum_{n' \in V(n)} w(n, n') = d(n) + 1$.

From item 5, it follows that if $d(n) = 2$, then the two edges $e_1, e_2 \in E$ connected to it have $(w(e_1), w(e_2)) \in \{(1, 2), (2, 1)\}$; and that if $d(n) = 3$, then the three edges $e_1, e_2, e_3 \in E$ connected to it are such that $(w(e_1), w(e_2), w(e_3)) \in \{(1, 1, 2), (1, 2, 1), (2, 1, 1)\}$.

A *soliton switch* is defined as a node n of degree $d(n) = 3$ (Fig. 1). This is also referred to in this work as a "non-deterministic branch" when a soliton is incoming along the weight-2 edge.

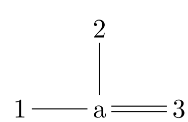

Figure 1: A soliton switch.

A *(partial) soliton path* of G is a sequence $n_0, n_1, ..., n_k$ for all $n_i \in N$ and $0 \le i \le k$ with $k > 0$ if:

1. n_0 is an exterior node;

2. $n_1, ..., n_{k-1}$ for $k > 1$ are interior nodes;

3. n_i and n_{i+1} share an edge; and

4. There exists a sequence of weighted graphs $G_0, G_1, ..., G_k$ such that:

   (a) $G_0 = G$;

   (b) For any $i = \{0, 1, ..., k-2\}$, G_{i+1} is defined if and only if $G_i = (N, E, w)$ is defined, and $|w_i(n_i, n_{i+1}) - w_i(n_{i+1}, n_{i+2})| = 1$; that is, the edge weights along the path differ by one; and

   (c) G_k is defined if and only if G_{k-1} is defined.

For the resultant graphs G_i and some $n, n' \in N$ with $(n, n') \in E$, $w_i(n, n') = w_{i-1}(n, n')$ only if $(n, n') \neq (n_{i-1}, n_i)$. Otherwise, $w_i(n, n') = 3 - w_{i-1}(n, n')$.

In other words, a path begins at an exterior node and consists of nodes linked by edges with weights differing by 1. As the soliton propagates along the path, each edge $e \in E$ it traverses changes its weight to $3 - w(e)$. In addition, a soliton may only traverse along a path of alternating weights.

The set $S(G)$ is defined as all possible G_k for some G.

A soliton path is *total* if n_k is an exterior node. The set $S(G, n, n')$ is the set of all possible weighted graphs G_k resulting from total soliton paths from n to n'.

A soliton graph G is a *chestnut* if it consists of a single cycle of even length, with paths leading into it that have an even length, and even spacings between the path entry points [2].

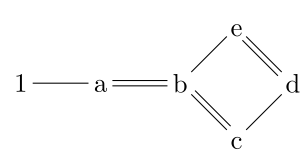

Figure 2: A chestnut.

2.2 Soliton Automata

An *alphabet* Σ is a non-empty set of symbols. A *word* or *string* is a finite sequence of symbols in an alphabet. A *semi-automaton* is a triple $A = (Q, \Sigma, \tau)$, where Q is a non-empty set, Σ is the input alphabet, and $\tau : Q \times \Sigma \to 2^Q$ is a map. The elements of Q are called *states*. τ is called the *transition function* of A. All automata we will consider in this work are semi-automata, and so we drop the "semi-" prefix.

If G is a soliton graph, and $X \in N$ is the set of exterior nodes of G, then the *soliton automaton* based on G is defined as $A(G) = (S(G), X \times X, \delta)$, where $S(G)$ is the set of states, $X \times X$ is the input alphabet, and $\delta : S(G) \times X \times X \to 2^{S(G)}$ is the transition function. We have that $\delta(G', n, n') = S(G', n, n')$ if $S(G', n, n') \neq \emptyset$; otherwise, $\delta(G', n, n') = \{G'\}$.

We say that $A(G)$ is *deterministic* if $|\delta(G', n, n')| \leq 1$ for all $G' \in S(G)$ and all exterior nodes $n, n' \in N$. It is *strongly* deterministic if there is exactly one soliton path from n to n' in G', for all $G' \in S(G)$ and exterior nodes $n, n' \in N$. It is *weakly* deterministic if it is deterministic but not strongly deterministic. For an example of weak determinism, see Appendix A.4.

Given a soliton automaton $A(G)$, G is a chestnut if and only if A is strongly deterministic in the single-soliton case [2].

2.3 Weighted Vectors

In order to arithmetize some examples, we introduce the concept of a *weighted vector* for a graph $G = (N, E, w)$. A weighted vector $(a - b)$ denotes the weighted edge between nodes $a, b \in N$ with $(a, b) \in E$ having weight 1, and a weighted vector $(a = b)$ denotes the same edge having weight 2. The complement of a weighted edge is formed by an edge of weight i undergoing a transformation to weight $3 - i$. In this vector, a is called the *start-node*, and b is called the *end-node*.

A *walk* in G is defined as a finite sequence of weighted vectors such that consecutive vectors v_1 and v_2 share exactly one node. Then the end-node of v_1 is the start-node of v_2. The first vector in a walk is a *walk-start* and the last vector is a *walk-end*. A *weighted walk* is a walk in which vector components are weighted edges of G.

A walk consisting of k nodes is denoted by $\mathbf{W} = (n_1, n_2, ..., n_k)$. A weighted walk is denoted by $\hat{\mathbf{W}} = (n_1 o_1 n_2 o_2 ... o_{k-1} n_k)$, where $o_i \in \{-, =\}$

for $1 \leq i < k$. All vectors in a walk with the same start- and end-nodes represent the same weighted edge in the soliton graph.

A soliton is a positional operator on a weighted walk. It *enters* the walk on the walk-start's start-node of the weighted walk, and acts by performing the complement operation on its vector and proceeding to the following vector in the walk. It continues until it has traversed the entire walk. If the walk-end's end-node is an exterior node, the walk is considered *total*, and the soliton exits upon complementing the walk-end. A soliton's position in a walk is indicated by $\bullet$ or $\circ$.

For some walk $\mathbf{W}$, we can define the formal polynomial $\mathbf{W}(x)$ as follows: a *term* x^{ab} represents a vector $(a, b) \in W$, and the formal sum of all terms in a walk is the edge polynomial $\mathbf{W}(x)$. We take $\oplus$ to be a commutative, associative, modulo-2 addition operation, with additive identity 0, as the formal sum operator. In addition, the factors in the exponent commute, i.e. $x^{ab} = x^{ba}$.

For example, a walk $\mathbf{W} = (1, a, b, c, a, 1)$ has the edge polynomial $\mathbf{W}(x) = x^{1a} \oplus x^{ab} \oplus x^{bc} \oplus x^{ca} \oplus x^{a1}$. Taking exponents to be commutative, we can indicate edges are undirected vectors. From the definition of $\oplus$, $\mathbf{W}(x)$ in its simplest form indicates edges that occur an odd number of times in a walk. In the above example, $\mathbf{W} = (1, a, b, c, a, 1)$, the simplest form of the edge polynomial is $\mathbf{W}(x) = x^{ab} \oplus x^{bc} \oplus x^{ca}$, and we see that the edges (a, b), (b, c), and (c, a) are traversed an odd number of times. In addition, $\mathbf{W}(x) = 0$ if and only if all edges are traversed an even number of times; i.e., the walk is an identity transformation on the soliton automaton.

A *subwalk* is a partial walk $\mathbf{W}'$ defined on a subsequence of consecutive nodes of the node sequence of $\mathbf{W}$. Subwalks inherit the arithmetic properties of terms.

A *weighted-edge polynomial* $\hat{\mathbf{W}}(x)$ represents a walk on a weighted graph $G = (N, E, w)$. Terms in $\hat{\mathbf{W}}(x)$ are of the form x^{ab} and $2x^{ab}$, where $a, b \in N$, $\{a, b\} \in E$, and $w \in \{1, 2\}$. x^{ab} corresponds to a vector $(a - b)$ and $2x^{ab}$ corresponds to a vector $(a = b)$. The coefficient represents the weight of the edge in the exponent of the term.

A *complement* $\mathbf{C}$ is an operation defined as $C(kt) = (3 - k)t$ for a term t and coefficient k.

2.4 Bursts

A *burst* b of length m is a word of the form $s_1||_{k_1}s_2||_{k_2}...s_{m-1}||_{k_{m-1}}s_m\bot$ with the following properties:

1. $m \in \mathbb{N}$;
2. $s_1, s_2, ..., s_m \in X$, where $X \subseteq N$ has the property that for any $x \in X$, $d(x) = 1$; and
3. $k_1, k_2, ..., k_{m-1} \in \mathbb{N}_0$.

Bursts are the input symbol to multi-wave soliton automata, and they model the time sequence of soliton injections. The term $||_k$ denotes a delay of k, so it follows that the term $s_i||_k$ denotes that soliton s_i is injected at time $t + \sum_{j=1}^{i} k_j$. If a burst starts at time t, then soliton s_1 is injected at time t, soliton s_2 is injected at time $t + k_1$, soliton s_3 is injected at time $t + k_1 + k_2$, and so on. The duration of a burst is denoted by $t(m) = \sum_{j=1}^{m-1} k_j$.

We illustrate automata in this paper via a physicochemical analogy. Weight-one edges are illustrated with a single line, and weight-two edges are illustrated with a double line. Letters denote interior nodes and numbers denote exterior nodes. The symbols $\bullet$, $\circ$ and $\odot$ indicate solitons.

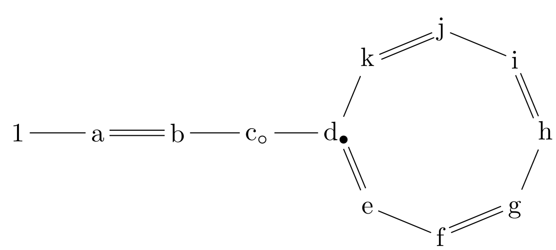

Figure 3: Two solitons in a soliton automaton. This automaton also happens to be a chestnut.

3 Identity

We will begin by examining the following question: can a multi-wave soliton automata execute a burst that acts as the identity operation?

Theorem 3.1. *Any even-length burst acts as the identity on a soliton graph* $G = (N, E, w)$ *with* $d(n) \leq 2$ *for any* $n \in N$*, and any odd-length burst does not.*

Proof. Since for all n, $d(n) \leq 2$, the graph takes the form of a straight line. The weight polynomial for such a configuration, for any soliton, will be of the form $\mathbf{W}(x) = x^{1s_1} \oplus x^{s_1 s_2} \oplus ... \oplus x^{s_{k-1} s_k}$, with $k = |N|$. There are an even number of solitons, so there are $2m\mathbf{W}(x)$ traversals for some non-zero m. $2\mathbf{W}(x) = 0$, so any even-length burst acts as the identity. □

Note that this is not necessarily the case with a tree (N, E), with $d(n) = 3$ for some $n \in N$. Consider the following soliton graph:

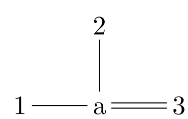

Figure 4: A soliton tree.

If a soliton begins at node 2, it proceeds to and exits via node 3.

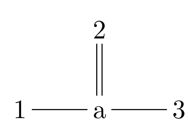

Figure 5: A soliton tree, after a first node.

If a second soliton is injected at node 2 after the first one in a burst, its behavior is non-deterministic and may choose either branch, exiting at nodes 1 or 3. If it chooses node 1, the burst is not the identity.

If a tree contains a node n with $d(n) = 3$, two of the three edges of n are of weight 1. This follows from the condition that the sum of the weights, on edges sharing a node, is equal to 4 for a node of degree 3 [2]. If a soliton encounters this node from a weight-2 edge, it has a non-deterministic choice (consider if a soliton were injected at node 3 in Figure 4). This non-determinism removes the possibility of a deterministic burst configuration that acts as the identity. As there are no impervious paths in the multi-wave automata case [1], multiple solitons will at some point be able to encounter this degree-3 soliton. Thus, the Theorem is not true if the graph contains a node of degree 3.

Lemma 3.1. *No chestnut with exactly one incoming path can have a burst of two solitons moving in parallel act as the identity.*

Proof. Let s_1 denote the first soliton in the soliton graph and s_2 denote the second. Let $n_0, n_1, \ldots, n_k$ be the total soliton path that each of the two solitons travel (it is identical for both). Let t_1 be the time the first soliton enters the graph of the automaton, and t_2 be the time the second soliton enters the graph. We know that all paths in a chestnut's cycle have an even length. We take l to be the length of the incoming path, and c to be the length of the cycle.

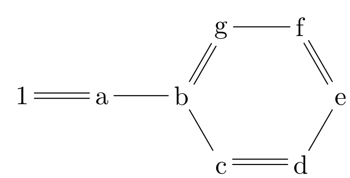

Figure 6: A simple chestnut, with $l = 2$ and $c = 6$. This graph is similar to the graphic of the chemical model of the Ethylbenzene molecule.

We first consider the case when $t_2 - t_1$ is odd.

Case 1: $t_2 - t_1 \geq 2l + c$

[1] This is proved later in Section 5.

In this case, the solitons are not moving in parallel. This trivially does not act as the identity in parallel.

Case 2: $l + c \leq t_2 - t_1 < 2l + c$

In this case, the first soliton has exited the cycle prior to the second one entering it. This will result in two solitons attempting to cross the same edge simultaneously, which is forbidden by the multi-wave soliton automata model. Then these paths do not exist.

Case 3: $c \leq t_2 - t_1 < l + c$

In this case, s_2 enters the graph when s_1 is within the cycle. If $l < c$, s_2 enters the cycle with s_1 still in it, and the case reduces to Case 4. Otherwise, the two solitons are directed towards each other on the incoming edge, and Case 2 is produced.

Case 4: $t_2 - t_1 < c$

In this case, the first soliton enters the cycle followed closely by the second one. Both exist in the cycle simultaneously. The parity of the weight of the first soliton's first edge in the cycle will differ from that of the second's: that is, if s_1 enters the cycle on an edge with $w(E) = n$, s_2 will enter the cycle on an edge with $w(E) = 3 - n$, as s_1 has inverted the weight on it. s_1 will arrive at a deterministic choice at the cycle (as this is a property of a chestnut [2]), whereas s_2 will be forced to follow it (anything otherwise would be a forbidden transition). Note that s_2 would have a non-deterministic choice if the solitons were not entering a cycle.

For a worked example, refer to Appendix A.1.

The first soliton will not be able to exit the cycle on the first iteration, as s_2 will have inverted the incoming edge into the cycle. It continues on the cycle. s_2, following closely, will exit the cycle and return to the exterior node. s_1 will continue its loop and exit.

The second soliton, s_2, has a non-deterministic choice to make. If it exits, the cycle is traversed three times - by s_1 twice and by s_2 once. As this is an odd number, the chestnut's cycle is inverted and thus in this case the soliton graph is different. However, if s_2 follows s_1, we find there is no change to the cycle (each soliton traverses it twice), and we have traversed the cycle a total of four times. Regardless of the number of times the second soliton follows the first, the final graph is the same. Eventually, the solitons exit and we arrive at the first outcome (with s_2 exiting). Thus, the cycle is traversed an odd number of times, and the burst is not the identity.

We must consider the case when $t_2 - t_1$ is even. This follows mostly the same as the odd case, except that two solitons may potentially proceed towards each other, as two solitons may simultaneously occupy a node. If the solitons progress in the same direction, we get the same result as the odd case. However, if they choose alternate directions, they both occupy the same node.

Before they occupy the same node, the edges they progress on will have weights w and $3 - w$. Upon occupying the node, the edges have weights $3 - w$ and w, respectively. This presents a conflict, as the first node will have entered on an edge with weight w. It cannot move backward and the only other edge now has weight w, and so the path does not exist. Then this case is not the identity.

As these cases comprehensively cover each possible timing, the proof is complete. □

Theorem 3.2. *No burst acts as the identity for chestnuts with exactly one incoming path.*

Proof. We will now generalize the result of Lemma 3.1. We take l to be the length of the incoming path, and c to be the length of the cycle. From Lemma 3.1, we reuse the first three cases: we assume all three solitons are in the automata simultaneously at some point, and that the burst's duration is less than c, as otherwise the solitons will definitely collide attempting to enter the cycle. The cases are similar to the length-2 burst above: the solitons will collide on the incoming path, or all simultaneously exist within the cycle.

For an example, see Appendix A.2.

Consider $\mathbf{W}_p(x) = x^{1p_1} \oplus ... \oplus x^{p_{k-1}r_1}$ to be the *incoming walk*, comprised of l terms, and $\mathbf{W}_r(x) = x^{r_1r_2} \oplus ... \oplus x^{r_{c-1}r_1}$ to be the *cyclic walk*, comprised of c terms, for a burst b. Both walks are identical for each soliton, so we refer to the shorthand $\mathbf{W}_p(x) = \mathbf{W}_p$, and $\mathbf{W}_r(x) = \mathbf{W}_r$. Assuming each soliton does exit the cycle at some point, every soliton's walk has $2\mathbf{W}_p$ as a subwalk.

In the even $|b|$ case, we find the first soliton will not exit and will contribute $2\mathbf{W}_r(x)$. The second soliton will have a non-deterministic choice. If it does not leave, an even number of cycles occur (as there are an even number of solitons in the cycle), so we discard the consideration of non-deterministic solitons cycling in this case. A third soliton will follow the second, and a fourth will have its own non-deterministic choice. Then the second soliton contributes $1\mathbf{W}_r(x)$ (as it immediately exits). The third contributes $1\mathbf{W}_r(x)$ as well. The fourth contributes $1\mathbf{W}_r(x)$ when it immediately exits, and so on. When the first soliton arrives back, it has contributed $2\mathbf{W}_r(x)$, and exits. Then the total walk sum is $\mathbf{W} = (2 + m)\mathbf{W}_r + 2|b|\mathbf{W}_p$, where $m = |b| - 1$. $\mathbf{W} \neq 0$, as $2 + m = 2 + |b| - 1$ is not even, so it is not the identity.

In the odd $|b|$ case, the first soliton exits after one cycle, contributing $1\mathbf{W}_r$. This is because $\mathbf{W}_p$ is traversed $|b|$ times, an odd number, and so the leading soliton will deterministically exit. The second soliton may proceed infinitely cyclically, but as there are now an even number of solitons propagating, any effect this weak determinism has is nullified by following solitons. When the next soliton exits, it has contributed $(1 + m)\mathbf{W}_r$ times, and is followed by the next soliton, itself having $(1 + m)\mathbf{W}_r$ cycles. This repeats for all solitons remaining. Then the total sum of the walks is $\mathbf{W} = |b|\mathbf{W}_p + (1 + j[(1 + m_1) + (1 + m_2) + ...])\mathbf{W}_r$, where $m_1, m_2, ...m_j$ are the number of weakly-deterministic cycles, and j is the (even) number of solitons that are not the first soliton. Because $1 + j[...]$ is odd, the burst is not the identity.

Because we have proved this for both even and odd burst lengths, any valid soliton bursts on chestnuts with one entry path are not the identity. □

For more complex soliton graphs, we simply need to prove there is some path for which a burst does not act as the identity. The following proofs will make use of this strategy.

Theorem 3.3. *No burst from one external node acts as the identity for chestnuts.*

Proof. In the case where there are multiple external nodes in a chestnut, the constraint on burst size is no longer sufficient. Solitons, in fact, may not follow their leading soliton, as they may exit via another incoming path before the solitons succeeding them reach the node they reside at. Some solitons may not cross their entry path twice (if they are to exit to a different external node).

We can generalize Theorem 3.2 for chestnuts with any number of incoming paths, assuming the solitons in the burst do not conflict. If they do conflict, the burst does not exist. Similarly to the single-incoming-path case, the first soliton will perform one cycle traversal immediately.

Assume a burst $s_1||_{k_1}s_2||_{k_2}...s_{m-1}||_{k_{m-1}}s_m\perp$ of length $m \geq 2$. s_2 has the option to potentially exit at any of the switches that are on the cycle but not connected to the entry path. If s_2 exits at the first available switch, s_1 completes its cycle and exits on the same path. If s_3 exists, it follows s_2. If s_4 exists, it now has a non-deterministic choice as well, and so on. Assume all solitons except s_1 in the burst non-deterministically exit on the first non-entry branch. Then the segment of the cycle not visited by s_2 through s_m will only have been visited once (by s_1), and so the parity of the number of traversals of this cycle segment will be odd. Thus, bursts are not the identity on chestnuts with multiple exit paths.

In tandem with Theorem 3.2, no burst from one external node acts as the identity for chestnuts. □

Theorem 3.4. *No burst acts as the identity for soliton automata on a graph composed of a cycle with exactly one incoming path.*

Proof. These structures are chestnuts, but with the possibility of cycles of an odd length. For even length cycles, Theorem 3.3 suffices. For odd-length cycles, the only difference is that an odd-length cyclic walk implies there are two similarly-weighted edges sharing a node in the cyclic walk.

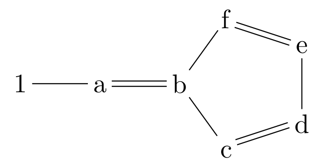

Figure 7: An odd-length cycle.

When a soliton encounters the incoming path, it will have a non-deterministic choice available. We know this as if the cycle has odd length, it will have two neighboring edges of identical weight. This segment would not be traversable unless the node shared by the two edges has a third edge. Any further solitons will follow it deterministically.

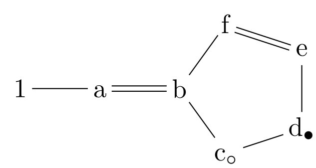

Figure 8: An odd-length cycle, with an even-length soliton burst performing a traversal.

Consider a graph with incoming path length l and cycle length c. If the burst length was even, the leading soliton exits after one traversal, and the odd-length remainder exit with weak determinism (and therefore do not act as the identity). If the burst length were odd, the leading soliton deterministically continues around the cycle after its first iteration, and the second has a weakly deterministic choice to make. If the second soliton exits, there are m cyclic path traversals, where m is the burst length $|b|$ less one (an even number). If the first soliton - with a weakly deterministic choice, finally - does not exit, there are $m + 2 + 1$ cyclic path traversals in total, an odd number.

We formalize this using weighted vectors. Consider $\mathbf{W}_p(x) = x^{1p_1} \oplus ... \oplus x^{p_{k-1}r_1}$ to be the *incoming walk*, comprised of l terms, and $\mathbf{W}_r(x) = x^{r_1 r_2} \oplus ... \oplus x^{r_{c-1}r_1}$ to be the *cyclic walk*, comprised of c terms, for a burst b. We use the shorthand $\mathbf{W}_p(x) = \mathbf{W}_p$, and $\mathbf{W}_r(x) = \mathbf{W}_r$. Assuming each soliton does exit the cycle at some point, every soliton's walk has $2\mathbf{W}_p$ as a subwalk.

In the even-length burst case, the solitons contribute $|b|\mathbf{W}_r$ as they cycle for the first time. The first soliton exits, contributing $\mathbf{W}_p$ deterministically. The next solitons will then either non-deterministically exit, contributing $(|b| - 1)\mathbf{W}_p$, and causing the burst to act as the identity, or they will not exit. If they do not exit, the total cyclic walk sum is $|b|\mathbf{W}_r + n(|b| - 1)\mathbf{W}_r$, for some n iterations. Assume n is odd. Then $|b| + n|b| - n = |b| + n(|b| - 1)$ is an odd number, and so this burst does not act as the identity in general.

In the odd-length burst case, the solitons again contribute $|b|\mathbf{W}_r$ as they cycle for the first time. However, the first soliton now deterministically cycles again. Its total contribution is $(2 + n)\mathbf{W}_r$, for some $n \geq 0$. If the following solitons exit non-deterministically, it then exits deterministically, and the total cyclic walk sum is $2\mathbf{W}_r + (|b| - 1)\mathbf{W}_r = (1 + |b|)\mathbf{W}_r$, an even number, and so acts as the identity. If they do not, however, the total cyclic walk length is $(2 + n)\mathbf{W}_r + n(|b| - 1)\mathbf{W}_r$. $(2 + n + |b|n - 1)\mathbf{W}_r$, for some $n \geq 0$. $2+n+|b|n-1 = 1+(1+|b|)n$, an odd number. Because $\mathbf{W}_r$ is an odd-length subwalk, this walk has an odd length and so is not the identity.

As this proves all cases, the proof is complete. □

Lemma 3.2. *No burst from one external node acts as the identity for soliton automata on a graph composed of a cycle with incoming paths of odd-length* *separation.*

Proof. The cases for even separation of paths and for exactly one path are proved above. In the odd-separation case, the first soliton no longer deterministically traverses the cycle, but instead non-deterministically may exit at an incoming branch.

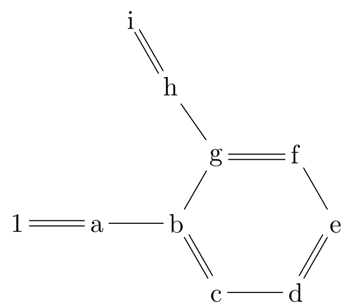

Figure 9: Odd-length separated cycle.

Assume the burst $s_1||_{k_1}s_2||_{k_2}...s_{m-1}||_{k_{m-1}}s_m\perp$. Upon reaching the odd-length separated path, assume s_1 does not exit. Then it, and s_2, deterministically proceed around the cycle, encountering the next switch. s_2 has a non-deterministic choice to exit or not, and we assume it and all its successors exit on the first available non-deterministic branch. We arrive at the same conclusion as Theorem 3.4. The first soliton has the ability to perform an odd number of cycle traversals, and if it does so, the burst is not the identity.

In an odd-length burst, we may simply allow the first soliton and all successors to exit on the first non-initial incoming path. Then the initial incoming path is traversed an odd number of times, and the burst does not act as the identity.

Then no burst acts as the identity for cycles with paths of odd-length separation. □

Theorem 3.5. *No burst from one external node acts as the identity for soliton automata on a graph composed of a cycle.*

Proof. This directly follows from the methodology of Theorem 3.3 being applied to odd-length cycles. Assume all solitons except the initial soliton immediately exit on the first available path that is not the entry path. Then the remainder of the cycle is traversed an odd number of times, and the burst is not the identity. In tandem with Theorem 3.3 and Lemma 3.2, all cases are covered and the proof is complete. □

4 Involution

A transformation that acts as the identity is defined as $f(x) = x$. Similarly, a transformation that is its own inverse is called an *involution*, i.e. an involution is defined as $f(f(x)) = x$. For a burst to be involuting, it must be applied to an automaton twice in succession, after the first of the two bursts have exited. If the parity of the number of times each edge is traversed by the first burst agrees with the parity of the number of times each edge is traversed by the second burst, it is considered an involuting burst.

Theorem 4.1. *Any burst is involuting on a soliton automata on a graph $G = (N, E, w)$ with $d(n) \leq 2$ for any $n \in N$.*

Proof. From Theorem 3.1, an even burst length results in the identity transformation I, and an odd burst length results in the complement of the identity transformation C, where $I(w) = w$ and $C(w) = 3 - w$. By definition, $I \times I = I$ and $C \times C = I$, and in both cases, the result of two bursts acts as the identity transformation. Then any burst is involuting on these graphs. □

Theorem 4.2. *Every soliton graph $G = (N, E, w)$ that has no cycles (i.e. a weighted tree) and that has a path through exclusively non-determinstic branches has an involuting burst.*

Proof. We proceed by cases for each node in G.

Case 1: $d(n) = 1$

n is an exterior node. Every soliton will traverse its edge at least once, and so the number of traversals in an involution is $2m$ for a burst length $m > 0$.

Case 2: $d(n) = 2$

n is connected in a line, and solitons cannot move backwards. Each edge in the path is visited, and there will be $2m$ traversals for some integer $m > 0$ after two bursts.

Case 3: $d(n) = 3$

Consider now a component with degree 3.

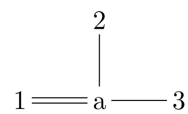

Figure 10: Case 3.

Because the soliton automaton is a tree, we are assured each soliton that enters the structure will only enter it once before exiting. The structure will by definition have one weight-2 edge and two weight-1 edges; the specific arrangement does not influence the outcome.

If a soliton enters the structure on the weight-2 edge, it will non-deterministically pick one of the weight-1 edges to exit on. When the burst is repeated, the soliton then deterministically follows the first, involuting the structure.

As this covers all cases, the structure is involutible. □

The condition of non-deterministic branching is required for the above proof as any other configuration would not be deterministic. Consider where there is a node $n \in N$ of $d(n) = 3$, where a soliton enters via an edge $e \in E$ with $w(e) = 1$. The soliton deterministically exits via the weight-2 edge. The same soliton entering the component now has a non-deterministic choice, and may not follow the first one, which causes the burst to not necessarily be involutible. In addition, if a longer burst occurs on the component, solitons incoming after the first soliton will also have a non-deterministic choice and the soliton may not involute.

Consider a cycle in a soliton graph with one incoming path. For an automaton of this form to be valid, a soliton must proceed down the incoming path, around the cycle some number of times, and exit.

Lemma 4.1. *All valid bursts are involuting on chestnuts with one exterior node.*

Proof. From Theorem 3.2, these bursts result in a definitively odd number of cyclic walks, regardless of length and timing, and therefore are involuting. □

The necessity of distinct cyclic walk counts is required for proofs regarding involution. Consider two identical bursts, where one has an odd amount of cyclic walks, and the other has an even amount, due to the effects of non-determinism. Then the resulting walks have an odd number of traversals over the edges in the cycle, and so the bursts would not be involuting. In any graph where a burst may or may not act as the identity, the burst is not involuting, as walks may traverse the cycle an even number of times in the first burst and an odd number of times in the second burst. In addition, if a burst may or may not visit some edges, it is not involuting.

Lemma 4.2. *No bursts from one exterior node are involuting on chestnuts with multiple exterior nodes.*

Proof. Assume a burst $s_1||_{k_1}s_2||_{k_2}...s_{m-1}||_{k_{m-1}}s_m\perp$ of length $m \geq 2$. s_2 has the option to potentially exit at any of the switches that are on the cycle but not connected to the entry path. s_1 will perform one cycle regardless of the remainder of the burst.

Let e_i be the number of solitons that traverse some branch i, where $i \geq 1$, and the initial incoming path count is e_1. $e_1 \geq m$, as the burst comes from one external node. We know that solitons will move in pairs, so $e_i \text{mod} 2 = 0$. Because they all move in pairs, the majority of solitons will not have any effect on the graph. Thus, we examine only the beginning and end nodes.

In an odd burst, s_1 traverses the cycle once and has weak determinism in the cycle. Then it is not involuting, as it may cycle an even or odd number of times before exiting. If the first burst cycles it an even number of times and the second burst cycles it an odd number of times, the total number of cyclic traversals is odd.

In an even-length burst, s_m has weak determinism and is followed by s_1. If s_m exits at a non-initial incoming path, the segment between it and the initial incoming path is visited an odd number of times (recall Theorem 3.3), and the other segment is visited an even number of times. If s_m exits at the incoming path, the entire cycle is visited an odd number of times. Because the parity of the two segments do not match, it is not involuting.

Then no burst from one exterior node is involuting on chestnuts with multiple exterior nodes. □

Lemma 4.3. *No bursts are involuting on soliton automata on graphs composed of a cycle with an odd length and an incoming path.*

Proof. From Theorem 3.4, these bursts may or may not act as the identity. Then the bursts are not involuting. □

Theorem 4.3. *No bursts from one exterior node are involuting soliton automata on graphs composed of a cycle, except chestnuts with one exterior node.*

Proof. Using Theorems 3.4 and 3.2, this is proved in an identical fashion to 4.3. In conjunction with 4.3, the proof is complete. □

5 Impervious Paths in Parallel

In the single-soliton case, an *impervious path* is a soliton path that is never traversed, regardless of which exterior node it is input from. In the multi-wave case, an impervious path is a soliton path that is never traversed, regardless of burst properties. If an impervious path exists in a soliton automaton, its transition monoid is identical to that of the automaton on its soliton graph, with the edges and nodes in the impervious path removed. If a soliton graph has an impervious path, it is *reducible*. If it does not have an impervious path, it is *reduced*.

Theorem 5.1. *No soliton automaton has an impervious path in the multi-wave model.*

Proof. We prove by cases. Let $d(n)$ be the degree of some node in a soliton graph. We use the property that $1 \leq d(n) \leq 3$ for soliton graphs.

Case 1: $d(n) = 1$

n is an exterior node. It has one path to travel, and so this node does not have any impervious edges.

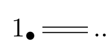

Figure 11: Case 1.

Case 2: $d(n) = 2$

The node is connected in a line (i.e. is reducible). In any soliton automata, solitons cannot move backwards. So, in a subautomata consisting of a line, each edge in the path is visited, and so no edge from this node is impervious.

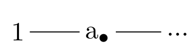

Figure 12: Case 2.

Case 3: $d(n) = 3$

In this case, a soliton traverses a switch. There is one edge $e \in E$ of weight $w(e) = 2$ and two edges $e \in E$ of weight $w(e) = 1$. Consider a burst $s_1||_1 s_2\bot$, where the second soliton comes immediately after the first. At this switch, if s_1 has a non-deterministic choice, it will non-deterministically choose one edge, meaning both edges may be visited and thus they are not impervious. If it has a deterministic choice, s_1 will traverse the deterministic edge, and s_2 now has a non-deterministic choice of which edge to visit. If it visits the edge s_1 did not visit, each edge has been visited, and there is no impervious path.

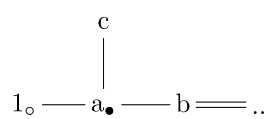

Figure 13: Case 3. s_1 enters from 1, followed by s_2.

Then, no node in a soliton graph has an impervious edge connected to it in the multi-wave case, and so no multi-wave soliton graph is impervious. □

Because there are no impervious paths in the multi-wave case, we arrive at the following corollary:

Corollary 5.1. *No multi-wave soliton graph is reducible.*

It is important to note that this does not indicate that for each soliton graph there is a burst that forces every edge to be traversed; only that there are a series of bursts that will traverse every edge. The non-deterministic behavior of solitons where $d(n) = 3$ for some $n \in N$ only implies there is no edge that cannot be traversed.

6 Determinism

Non-determinism is present whenever a soliton encounters a switch with two equal-weighted potential edges. If at least one of the two equal-weighted potential edges eventually reaches the same state, i.e. if at least one of the two potential paths is a cycle, the automaton is weakly deterministic. The most obvious difference between weak determinism and strong determinism is that weakly deterministic automata do not have an upper limit on path length.

Lemma 6.1. *If there is a cycle in a soliton graph, the associated soliton automaton is not strongly deterministic.*

Proof. Consider some burst, such that at least 2 solitons approach the cycle (if a cycle exists, it will be traversed by at least some of the burst, from Theorem 5.1). If a following soliton proceeds along the cycle in an opposite direction to the leading soliton, the two will eventually meet causing conflict, and so this path doesn't exist. So, the only path to take is to follow the first soliton. This causes all solitons to propagate around the cycle. If the first soliton has the conditions to exit deterministically (i.e. a switch with two different weights available to the soliton), it does so, causing the edge it exits on to be identical to the edge in the cycle. A second soliton will then encounter a non-deterministic switch. Otherwise, the first soliton encounters a non-deterministic switch. Therefore, the soliton automaton is not strongly deterministic. □

For an example, refer to Appendix A.3. We can generalize this to all switches in soliton automata:

Theorem 6.1. *The presence of a switch implies that a soliton automaton is non-deterministic in the multi-wave case.*

Proof. Recall Theorem 5.1. Consider a non-deterministic switch (two edges with the same weight and a third of a different weight), with a soliton burst of some length such that it is traversed. If there are no impervious paths in a multi-wave automaton, each connected component of the soliton graph could be visited, so the determinism of the graph depends on the determinism of the components within it - such as this switch. If this switch is non-deterministic, so is the automaton. When a soliton enters a non-deterministic switch (necessary for some burst, as there are no impervious paths), it either has a non-deterministic choice of which path to take, or takes a deterministic path, “rotating” the switch into a non-deterministic configuration for any soliton following it. Then the switch is non-deterministic, as is the automaton. □

Theorem 6.2. *All multi-wave soliton automata are not strongly deterministic, except the trivial case of a graph with maximum degree* 2.

Proof. All multi-wave soliton automata with a switch are non-deterministic (Theorem 6.1). The only soliton automata without switches are paths of degree ≤ 2. Each soliton linearly propagates from one exterior node to another, and so these automata are strongly deterministic. □

7 Conclusion

In this thesis, we examined various results from the definition of the multi-wave soliton model. We have proved that there are no reducible soliton automata in the multi-wave case, as there are no impervious paths. We have also proved that no non-trivial multi-wave soliton automaton (i.e. with maximum degree greater than 2) is strongly deterministic. The results in this paper generate even more questions: what possible alterations of the model can be made to allow for parallel computation, despite the lack of strong determinism in most soliton automata? Can the model be modified to encourage an upper time bound on weak determinism? Does the choice between initial exterior nodes in the multi-wave case matter, or if bursts originate from a combination of various exterior nodes? These questions should be addressed in a future, more in-depth work.

References

[1] H. Bordihn, H. Jürgensen, H. Ritter: Multi-wave soliton automata. *Theoret. Comput. Sci.* to appear.

[2] J. Dassow, H. Jürgensen: Soliton automata. *J. Comput. System Sci.* **40** (1990), 158–181.

[3] J. Dassow, H. Jürgensen: Deterministic soliton automata with a single exterior node. *Theoret. Comput. Sci.* **84** (1991), 281–292.

[4] J. Dassow, H. Jürgensen: Deterministic soliton automata with at most one cycle. *J. Comput. System Sci.* **46** (1993), 155–197.

[5] J. Dassow, H. Jürgensen: The transition monoids of soliton trees. In G. Păun (editor): *Mathematical Linguistics and Related Topics, Papers in Honour of Solomon Marcus on His 70th Birthday*. 76–87, Editura Academiei Române, Bucureşti, 1995.

[6] A. S. Davydov: *Solitons in Molecular Systems*. D. Reidel Publishing Company, 1985.

[7] H. Jürgensen, P. Kraak. Transition monoids of soliton trees, 2007. Manuscript.

[8] M. Manciu, S. Sen, A. J. Hurd: Crossing of identical solitary waves in a chain of elastic beads. *Physical Review E* **63**(1) (2000), 016614–1–6.

Appendices

A Examples

A.1 Example 1

Two solitons, s_1 and s_2, are to propagate through a chestnut. First, we consider if there is an odd distance between the solitons in the burst:

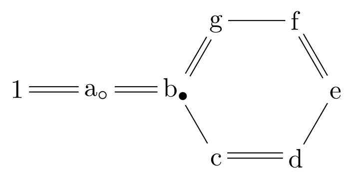

Figure 14: The simple chestnut, with $l = 2$ and $c = 6$. Both solitons have entered from 1, with $t_1 = 0$ and $t_2 = 1$. It is currently $t = 2$. We denote s_1 as $\bullet$ and s_2 as $\circ$.

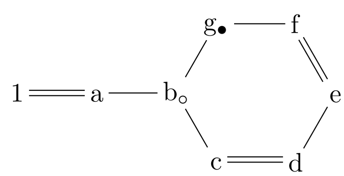

Figure 15: Above, after one time step. s_1 will be followed by s_2.

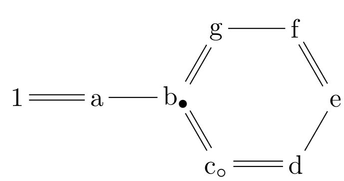

Figure 16: s_1 arrives at the entry point. It cannot exit, as it must traverse a weight 2 edge.

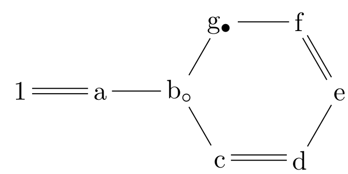

Figure 17: s_2 at a non-deterministic choice.

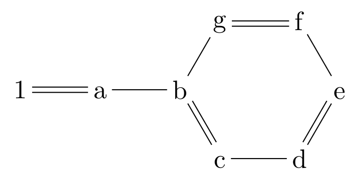

Figure 18: The solitons exit. The final soliton graph is produced.

We then consider the case where two solitons have an even-length spacing between them:

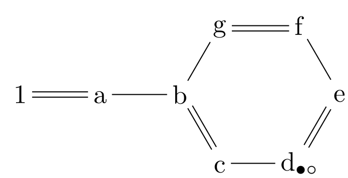

Figure 19: s_1 and s_2 coincide. They cannot move.

A.2 Example 2

Consider a burst of length 3. We take $l = 2$ to be the length of the incoming path, and $c = 6$ to be the length of the cycle. We will use the burst $s_1||_1 s_2||_1 s_3\bot$.

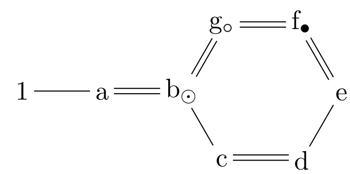

Figure 20: The chestnut, with three solitons s_1, s_2, and s_3, denoted by $\bullet$, $\circ$, and $\odot$.

When they enter the cycle, we proceed as above in the case of two solitons. Due to the odd number of solitons, however, s_1 will be able to exit on its first pass.

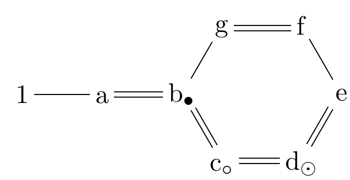

Figure 21: s_1 prepares to exit.

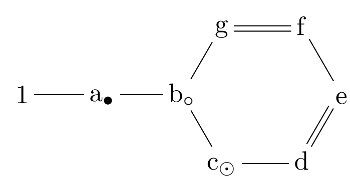

Figure 22: s_1 exits.

We can now see s_2 has a non-deterministic choice, similarly to the 2-soliton case. If s_2 exits, s_3 will follow it and exit. This results in the cycle being traversed three times (an odd number), and so the identity has not been applied to the cycle. The incoming path has an even traversal (6 times). However, if s_2 does not exit, the automaton becomes extremely similar to the 2-soliton case.

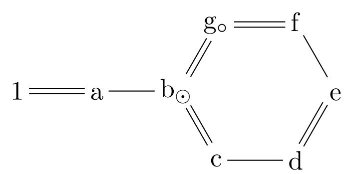

Figure 23: s_2 does not exit.

The cycle is traversed once by each of s_2 and s_3. We return to Figure 22, and make another non-deterministic choice. Regardless, we have cycled an odd number of times when the burst finally completes, and so the burst is not the identity.

A.3 Example 3

Consider the following soliton graph, with the burst $s_1||_1 s_2\bot$:

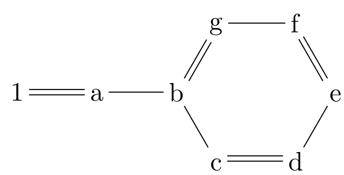

Figure 24: A simple soliton graph with a cycle.

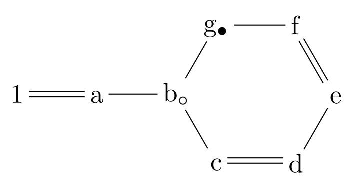

Figure 25: The soliton graph after 2 time steps.

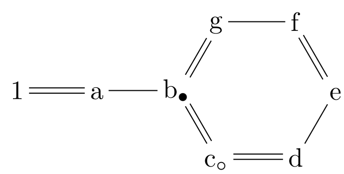

Figure 26: The soliton graph after 5 more time steps.

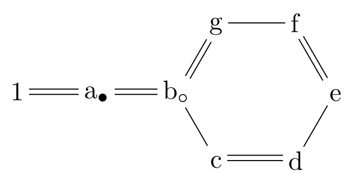

Figure 27: The following soliton graph.

At this point, s_2 has a non-deterministic choice, and the graph is not strongly deterministic. Exiting will leave the cycle's edges as-is, and performing another cycle will invert the edge weights in the cycle. Consider the same graph with the burst $s_1||_1 s_2||_1 s_3\perp$:

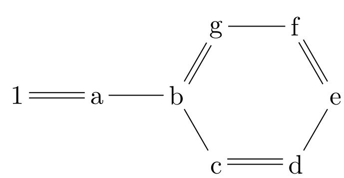

Figure 28: The simple soliton graph.

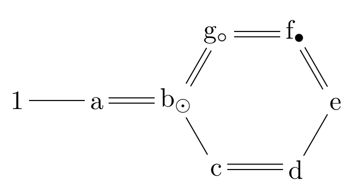

Figure 29: The burst, propagated along the graph.

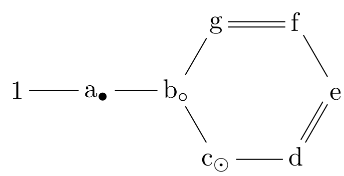

Figure 30: A following soliton graph.

Once again, s_2 has a non-deterministic choice and so the graph is not deterministic. If s_2 exits, s_3 must follow, and the final soliton graph is:

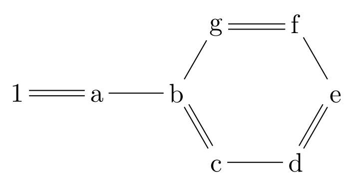

Figure 31: One final soliton graph.

Otherwise, the two remaining solitons propagate once and we find the following graph:

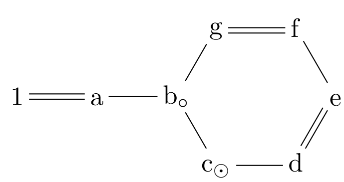

Figure 32: Weakly deterministic soliton graph.

This graph is identical to the prior graph, and so this burst is weakly deterministic (as it can repeat infinitely).

A.4 Example 4

We will demonstrate the behavior of weak determinism here.

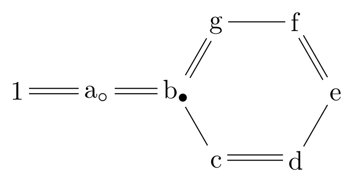

Figure 33: Chestnut, $l = 2$ and $c = 6$. Both solitons have entered from 1, with $t_1 = 0$ and $t_2 = 1$. It is currently $t = 2$. We denote s_1 as $\bullet$ and s_2 as $\circ$.

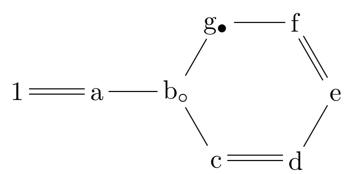

Figure 34: s_1 moves to g. s_2 moves to b.

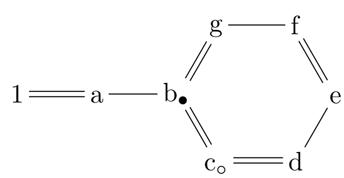

Figure 35: s_1 and s_2 have traversed the cycle. s_1 now proceeds to node g again.

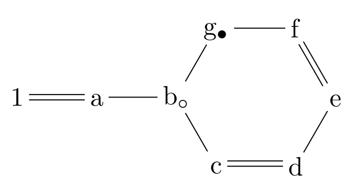

Figure 36: s_2 now has a non-deterministic choice of remaining in the cycle or not.

If s_2 remains in the cycle, we find that the graph and possible soliton transitions are identical to that of Figure 34. Figure 36 is reached again.

This can continue infinitely. However, if s_2 does not remain in the cycle, it will always exit on the incoming path. This is weak determinism as when (if) s_2 exits the cycle, it has the same behavior as if it had left the cycle at an earlier or later time.